We throw m balls uniformly randomly into h bins B_1, B_2, ..., B_h, each of which contains k cells, so that each cell contains at most one ball. Let the “load factor" $\alpha \equiv \frac{m}{hk} \leq \frac{1}{2}$. Let X_1, X_2, ..., X_h be the number of balls, among the total of m balls thrown, that falls into bins B_1, B_2, ..., B_h respectively. Let Y_1, Y_2, ..., Y_h be i.i.d. RVs distributed as $Binomial(k, \alpha)$. Let $f(x_1, x_2, ..., x_h)$ be any non-negative, monotonically increasing or decreasing function. Then we have the following theorem:

Theorem

$$\mathrm{E}[f(X_1, X_2, ..., X_h)] \leq 2\mathrm{E}[f(Y_1, Y_2, ..., Y_h)]$$

Propositions needed to prove the Times-2 Bounds

Let $X_i^{(l)}$ have the same semantics as X_i, , $i = 1, 2, ..., n$, except that l instead m balls are thrown. Then we have:

Proposition

$\mu(X_1^{(l)}, X_2^{(l)}, \ldots, X_h^{(l)}) = \mu(Y_1, Y_2, \ldots, Y_h | \sum_{j=1}^h Y_j = l)$ where $\mu(Z)$ denotes the distribution of a random variable or vector Z.

In other words, conditioned upon $\sum_{j=1}^h Y_j^{(\alpha)} = l$, the independent random variables $Y_1, Y_2, \ldots, Y_h$ have the same joint distribution as dependent random variables $X_1^{(l)}, X_2^{(l)}, \ldots, X_h^{(l)}$.

Proposition

For any $0 \leq l < l' \leq N \equiv hk$, we have $[X_1^{(l)}, X_2^{(l)}, \ldots, X_h^{(l)}] \leq_{st} [X_1^{(l')}, X_2^{(l')}, \ldots, X_h^{(l')}]$.

Stochastic ordering

Definition (Stochastic order (Stoyan 2002))

The random variable X is said to be smaller than the random variable Y in stochastic order (written $X \leq_{st} Y$), if $\Pr[X > t] \leq \Pr[Y > t]$ for all real t, or equivalently, if $\mathrm{E}[f(X)] \leq \mathrm{E}[f(Y)]$ holds for all increasing functions f, for which both expectations exist. If X and Y are random vectors, however, only the latter definition applies.

Proof

For increasing function f,

$$
\begin{aligned}
& E[f(Y_1, ..., Y_h)] \\
= \; & \sum_{l=0}^{N} E[f(Y_1, ..., Y_h) \mid \sum_{j=1}^{h} Y_j = l] \Pr[\sum_{j=1}^{h} Y_j = l] \\
\geq \; & \sum_{l=m}^{N} E[f(Y_1, ..., Y_h) \mid \sum_{j=1}^{h} Y_j = l] \Pr[\sum_{j=1}^{h} Y_j^{(\alpha)} = l] \\
= \; & \sum_{l=m}^{N} E[f(X_1^{(l)}, ..., X_h^{(l)})] \Pr[\sum_{j=1}^{h} Y_j^{(\alpha)} = l] \\
\geq \; & \sum_{l=m}^{N} E[f(X_1^{(m)}, ..., X_h^{(m)})] \Pr[\sum_{j=1}^{h} Y_j = l]
\end{aligned}
$$

Proof cont'd

$$
\begin{aligned}
&= E[f(X_1^{(m)}, \cdots, X_h^{(m)})]\,\Pr[\sum_{j=1}^{h} Y_j \geq m] \\
&= E[f(X_1^{(m)}, \cdots, X_h^{(m)})] Binotail_{(\geq)}(hk, \alpha, m) \\
&\geq \frac{1}{2} E[f(X_1^{(m)}, ..., X_h^{(m)})] \\
&\equiv \frac{1}{2} E[f(X_1, ..., X_h)]
\end{aligned}
$$

The pivotal property of Binomial distribution $B(n, p)$, where $k = np$ is an integer:

- $Binotail_{(\geq)}(n, p, k) \geq \frac{1}{2}$
- $Binohead_{(\leq)}(n, p, k) \geq \frac{1}{2}$

Times-2 Bounds for Balls into Bins Model [Mitzenmacher and Upfal]

We throw m balls uniformly randomly into h bins B_1, B_2, ..., B_h. The “load factor" is defined as $\alpha \equiv \frac{m}{h}$. Let X_1, X_2, ..., X_h be the number of balls, among the total of m balls thrown, that falls into bins B_1, B_2, ..., B_h respectively. Let Y_1, Y_2, ..., Y_h be i.i.d. RVs distributed as $Poisson(\alpha)$. Let $f(x_1, x_2, ..., x_h)$ be any non-negative, monotonically increasing or decreasing function. Then we have the following theorem:

Theorem

$$\mathrm{E}[f(X_1, X_2, ..., X_h)] \leq 2\mathrm{E}[f(Y_1, Y_2, ..., Y_h)]$$

Proof involves similar propositions and the pivotal property of Poisson distributions.

Open question: Are there other such time-2 bounds?